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ON THE SPECTRAL TYPE OF ORNSTEIN'S CLASS ONE TRANSFORMATIONS

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ABSTRACT

It is shown that D. Ornstein's construction of class one transformations leads almost surely to singular maximal spectral type. Some related harmonic analysis problems are discussed.

1. Introduction

This Note is concerned with the spectral analysis of a class of transformations introduced by D. Ornstein in [5]. These transformations, called class one or rank one, have spectral multiplicity one. It is apparently an unsolved problem whether the spectral type may be absolutely continuous with respect to Lebesgue measure (such an example would be very meaningful to Banach's well-known problem whether a dynamical system (Ω, μ, T) may have simple Lebesgue spectrum). In [5] an example of a **mixing** rank one transformation is obtained. The construction makes essential use of probabilistic techniques. It is our purpose here to show that for such a random construction the maximal spectral type is always singular. This investigation will be mainly oriented towards harmonic analysis. Next we briefly recall Ornstein's construction. The dynamical system (DS) is defined on [0,1] with Lebesgue measure and is obtained as a limit of following process: At stage n, we have disjoint intervals $J_1, J_2, \dots, J_{h(n)}, J'$ partitioning [0,1]. The J_i $(1 \leq i \leq (h(n))$ are of the same length and T maps J_i linearly on J_{i+1} for $1 \leq i < h(n), J_{h(n)}$ is mapped in J'. The transformation T is not defined on J' at this stage. Points of J' will be mapped either in J' or in J_1. To describe the transformation at state $n+1$, fix an integer

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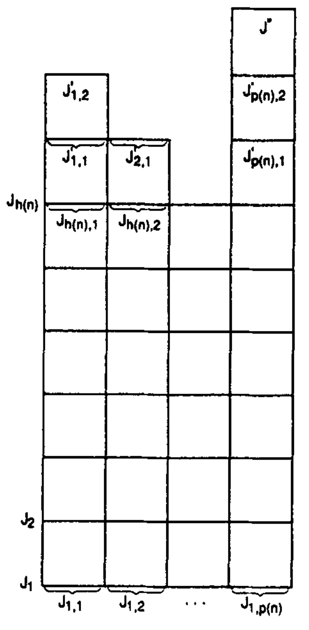
$p(n)$ and a sequence of integers $\{a(n,p) | 1 \leq p \leq p(n)\}$. Partition each J_i in equal length intervals $J_{i,p}$ $(1 \leq p \leq p(n))$ such that T maps $J_{i,p}$ linearly to $J_{i+1,p}$ $(1 \leq i < h(n))$. Partition J' in intervals $(J'_{p,a})_{\substack{1 \leq p \leq p(n) \\ 1 \leq a \leq a(n,p)}}$ and a remainder interval J''. For $1 \leq p \leq p(n)$, T maps $J_{h(n),p}$ linearly on $J'_{p,1}$ and $J'_{p,a}$ linearly on $J'_{p,a+1}$ for $a < a(n,p)$. The interval $J'_{p,a(n,p)}$ is mapped linearly on $J_{1,p+1}$ for $p < p(n)$. T maps $J'_{p(n),a(n,p(n))}$ into J'' and is not defined on J''. The intervals at stage $n+1$ are thus

$$\{J_{i,p} | 1 \leq i \leq h(n), 1 \leq p \leq p(n)\} \cup \{J'_{p,a} | 1 \leq p \leq p(n), 1 \leq a \leq a(n,p)\} \cup \{J''\}$$

and

$$h(n+1) = h(n) \cdot p(n) + \sum_{1}^{p(n)} a(n,p). \tag{1.1}$$



In Ornstein's construction, the $p(n)$ are rapidly increasing and the numbers $a(n,p)$ $(1 \leq p \leq p(n))$ chosen "at random" in a certain way (subject to certain bounds). This may be organized in various ways. Ornstein's procedure consists in choosing numbers $s(n,p)$ randomly in an interval $\{1,2,\ldots,K_n\}$, the selections being independent for different p. Here $K_n < h(n-1)$ and $K_n \to \infty$ for $n \to \infty$. One then lets $a(n,p) = h(n-1) + s(n,p) - s(n,p-1)$ with $s(n,0) = 0$.

But the argument presented in the next section applies equally well to different presentations.

2. Estimates on spectral measures

Given a function $f \in L^2(0,1)$, the corresponding spectral measure ν_f is defined by

$$\hat{\nu}_f(n) = \int_0^1 e^{-2\pi i n\theta}\nu_f(d\theta) = \langle f, T^n f\rangle \quad (n \in \mathbb{Z}). \tag{2.1}$$

Also, ν_f is almost surely the ω^*-limit in $M(\pi)$ (Radon measures on the circle) of the sequence of trigonometric polynomials

$$\frac{1}{N}\Big|\sum_{j=1}^{N} f(T^j\omega)e^{-2\pi i j\theta}\Big|^2 \tag{2.2}$$

(T is assumed ergodic).

The statement made in the introduction is the following.

PROPOSITION 1: *In the construction of [5], ν_f is always singular with respect to Lebesgue measure.*

In proving this, it clearly suffices to consider functions f that are "elementary" in the sense that for some integer n, f is constant on the intervals $J_1, \ldots, J_{h(n)}, J'$ appearing at state n. From the preceding, we have to analyze the behavior of the trigonometric polynomials

$$\frac{1}{\sqrt{N}}\left[\sum_{j=1}^{N} f(T^j\omega)e^{2\pi i j\theta}\right] \tag{2.3}$$

for $N \to \infty$.

Choose $m >> n$ and denote $v_j^{(m)}$ the value of f on the interval $J_j^{(m)}$ of the mth-partition. For $\omega \in J_1^{(m)}$ and $N = h(m)$, one gets from the dynamics of T

$\frac{1}{\sqrt{N}}\sum_{j=1}^N f(T^j\omega)e^{2\pi ij\theta}$

$$= \frac{1}{\sqrt{N}}\sum_{j=1}^{N} v_j^{(m)} e^{2\pi ij\theta}$$

$$= \left[h(m-1)^{-\frac{1}{2}}\sum_{j=1}^{h(m-1)} v_j^{(m)} e^{2\pi ij\theta}\right]$$

$$\times\left[p(m-1)^{-\frac{1}{2}}\left(1+\sum_{p=1}^{p(m-1)-1} e^{2\pi i(p(h(m-1)+h(m-2))+s(m-1,p))\theta}\right)\right] \tag{2.4}$$

$$+q_m \tag{2.5}$$

where q_m corresponds to the intervals

$$\{J'_{p,a} | 1 \le p \le p(m-1), 1 \le a \le a(m-1,p)\}.$$

Hence from (1.1)

$$\|q_m\|_2^2 \sim \frac{1}{N}\sum_{1}^{p(m-1)} a(m-1,p) < 2\frac{h(m-2)p(m-1)}{h(m)} < \frac{2}{p(m-2)}. \tag{2.6}$$

Iterating the preceding up to stage n, it follows that

$$(2.4) =$$

(2.7)

$$\left[h(n)^{-\frac{1}{2}}\sum_{j=1}^{h(n)} v_j^{(n)} e^{2\pi ij\theta}\right]\prod_{\ell=n+1}^{m-1}\left[p(\ell)^{-\frac{1}{2}}\left(1+\sum_{p=1}^{p(\ell)-1} e^{2\pi i(p(h(\ell)+h(\ell-1))+s(\ell,p))\theta}\right)\right]$$

$$+q_m+q_{m-1}+\cdots+q_n. \tag{2.8}$$

The L^1-norm of (2.8) is estimated by the L^2-norm, hence from (2.6), by

$$(\|q_m\|_2^2+\|q_{m-1}\|_2^2+\cdots+\|q_n\|_2^2)^{\frac{1}{2}} < 2\left(\frac{1}{p(n-2)}+\frac{1}{p(n-1)}+\cdots+\frac{1}{p(m-2)}\right)^{\frac{1}{2}}$$

$$\underset{n\to\infty}{\longrightarrow} 0. \tag{2.9}$$

Define for simplicity

$$(2.10) \qquad P_\ell(\theta) = p(\ell)^{-\frac{1}{2}} \sum_{p=0}^{p(\ell)-1} e^{2\pi i (p(h(\ell)+h(\ell-1)+s(\ell,p))\theta)} \quad (s(\ell,0)=0).$$

For general $\omega \in [0,1]$, (2.4) appears in the form

$$(2.11) \qquad \left[h(n)^{-\frac{1}{2}} \sum_{j=1}^{h(n)} v_j^{(n)} e^{2\pi i j\theta} \right] \cdot \prod_{\ell=n+1}^{m-2} P_\ell(\theta) \cdot Q_m(\theta) + R_m(\theta)$$

where $Q(\theta)$ is an L^2-normalized sum of at most $p(m-1)$ characters with $h(m-1)$-separated frequences and

$$\|R(\theta)\|_2 \leq \left[\frac{h(m-1)}{h(m)}\right]^{\frac{1}{2}} + (2.9) \to 0 \quad \text{for } n \to \infty.$$

Squaring the expression (2.11) and passing to ω^*-limits, it appears that for any fixed $\bar{n} > n$ one may write the spectral measure ν_f in the form

$$(2.12) \qquad \nu_f = h(n)^{-1} \left| \sum_{j=1}^{h(n)} v_j^{(n)} e^{2\pi i j\theta} \right|^2 \cdot \prod_{n<\ell\leq\bar{n}} |P_\ell|^2 \cdot \mu_{\bar{n}} + \nu_n$$

where $\mu_{\bar{n}}, \nu_n$ are measures on Π with $\|\mu_{\bar{n}}\| \leq 1$ and $\|\nu_n\| \to 0$ for $n \to \infty$.

It is clear from (2.12) that in order to prove $\nu_f \perp$ Lebesgue measure, it suffices to show that

$$(2.13) \qquad \int_\Pi \left| \prod_{\ell=n}^{\bar{n}} P_\ell(\theta) \right| d\theta \to 0 \quad \text{for } \bar{n} \to \infty.$$

This will be achieved next. We will take $n = 1$. The nature of the argument is such that the other cases are covered as well.

Remark: The previous considerations permit, in fact, to show that the maximal spectral type of T is absolutely continuous with respect to the measure

$$(2.14) \qquad \sum_{n\geq 1} 2^{-n} \prod_{\ell=n}^{\infty} |P_\ell|^2.$$

The frequencies of a given polynomial P_ℓ appear as random perturbation by the numbers $s(\ell,p)$ of the progression $p(h(\ell)+h(\ell-1))$, $p = 0,1,\ldots,p(\ell)-1$. In establishing (2.13), this randomization in $s(\ell,p)$ will be of importance.

Take a rapidly increasing sequence of integers $\mathcal{N}$ and estimate

$$\left\|\prod P_n\right\|_{L^1(\prod)} \leq \left\|\prod_{n\in\mathcal{N}} P_n\right\|_1 \cdot \left\|\prod_{n\in\mathcal{N}} P_n \cdot \prod_{n\notin\mathcal{N}} |P_n|^2\right\|_1 \tag{2.15}$$

where the second factor of (2.15) is bounded by $\|\Pi_n |P_n|^2\|_1 \leq 1$, since

$$\left\|\prod_{n\notin\mathcal{N}} |P_n|^2\right\|_1 \leq 1.$$

Thus (2.13) may be derived from the corresponding statement where the index ℓ is restricted to some subsequence $\mathcal{N}$. The point of this construction is to guarantee enough spectral dissociation of the factors in the product. In what follows, we will assume implicitly $n \in \mathcal{N}$. Denote $s(n)$ the sequence $\{s(n,p) | 1 \leq p \leq p(\ell)-1\}$ taken (at random) in the product

$$\Omega_n = \{1, 2, \ldots, K_n\}^{p(\ell)-1} \tag{2.16}$$

equipped with normalized counting measure. Since P_n depends on this sequence, we write $P_n = P_n(\theta, s(n))$. We will show that

$$\int_{\otimes\Omega_n} \int_{\Pi} \left| \prod_{n\leq m} P_n(\theta, s(n)) \right| d\theta ds \overset{m\to\infty}{\longrightarrow} 0 \tag{2.17}$$

(s denoting the product space variable).

Denoting $P = P_m$ and $Q = \Pi_{n<m} P_n$, one has the following:

$$\int |Q|\, \big| |P|^2 - 1 \big| \leq \left(\int |Q|\, \big| |P| - 1 \big|^2 \right)^{\frac12} \left(\int |Q| (|P| + 1)^2 \right)^{\frac12}. \tag{2.18}$$

The second factor in (2.18) is at most $(\int |Q|\, |P|^2)^{\frac12} + (\int |Q|)^{\frac12} < 2$. For the first factor

$$\int |Q|\, \big| |P| - 1 \big|^2 = \int |Q|\, |P|^2 + \int |Q| - 2 \int |Q|\, |P|, \tag{2.19}$$

$$\int |Q|\, |P|^2 \leq \int |Q| \tag{2.20}$$

(because the spectrum of P is sufficiently dissociated).

Hence, from (2.18), (2.19), (2.20)

$$\int |Q|\,\big|\,|P|^2-1\big| \le c\left(\int |Q| - \int |Q|\,|P|\right)^{\frac{1}{2}} \tag{2.21}$$

and therefore

$$\int \Big|\prod_{n<m} P_n \cdot P_m\Big| \le \int \Big|\prod_{n<m} P_n\Big| - c\left(\int \Big|\prod_{n<m} P_n\Big|\,\big|\,|P_m|^2-1\big|\right)^2. \tag{2.22}$$

In the preceding, $\int$ refers to integration on $\Pi \times \otimes\Omega_n$.

Write

$$|P_m|^2 - 1 = \frac{1}{p(m)} \sum_{p\neq q} e^{2\pi i[(p-q)(h(m)+h(m-1))]\theta}\, e^{2\pi i s(m,p)\theta}\, e^{-2\pi i s(m,q)\theta} \tag{2.23}$$

which absolute value we integrate on Ω_m, for fixed θ.

Consider the independent variables $(p<p(m))$ σ_p, σ_p' on $\{1,2,\dots,k_m\}$ defined as

$$\sigma(s) = e^{2\pi i s\theta}, \tag{2.24}$$

$$\sigma' = \sigma - \frac{1}{k}\sum_{1}^{k} \sigma(s). \tag{2.25}$$

Write

$$\sum_{p,q} a_{pq}\sigma_p\bar{\sigma}_q = \left(\sum a_{pq}\right)\Big|\int \sigma\Big|^2 + \sum a_{pq}\left(\int \bar{\sigma}\cdot\sigma_p' + \int \sigma\cdot\bar{\sigma}_q'\right) + \Sigma a_{pq}\sigma_p'\bar{\sigma}_q'. \tag{2.26}$$

Consider a random sign $\epsilon = (\epsilon_1,\dots,\epsilon_{p(m)-1}) \in \{1,-1\}^{p(m)-1}$. Taking conditional expectation of (2.26) with respect to the variables (thus the σ_p) for which $\epsilon_p = 1$, one finds the expression

$$\begin{aligned}&\left(\sum a_{pq}\right)\Big|\int \sigma\Big|^2 + \sum a_{pq}\left(\frac{1+\epsilon_p}{2}\cdot\int\bar{\sigma}\cdot\sigma_p' + \frac{1+\epsilon_q}{2}\cdot\int \sigma\cdot\bar{\sigma}_q'\right)\\ &\quad + \sum a_{pq}\frac{1+\epsilon_p}{2}\cdot\frac{1+\epsilon_q}{2}\sigma_p'\cdot\,\bar{\sigma}_q'.\end{aligned} \tag{2.27}$$

Integrating the absolute value of this expression on the product $\Omega_m \times \{1,-1\}^{p(m)-1}$, it follows from elementary harmonic analysis on the Cantor group* that there is a minoration by

$$c\int_{\Omega_m}\Big(\sum|a_{pq}|^2\,|\sigma'_p|^2\,|\sigma'_q|^2\Big)^{\frac12} > c\Big(\sum|a_{pq}|^2\Big)^{\frac12}\Big(\int|\sigma'|\Big)^2 \tag{2.28}$$

(performing first the integration in ε). Consequently one has for (2.23)

$$\int_{\Omega_m}|1-|P_m|^2| > c\left(1-\frac{1}{k_m}\Big|\sum_{s=1}^{k_m}e^{2\pi i s\theta}\Big|\right)^2. \tag{2.29}$$

Thus, by (2.22), (2.29)

(2.30)

$$\int\Big|\prod_{n<m}P_n\cdot P_m\Big| \le \int\Big|\prod_{n<m}P_n\Big| - c\left\{\int\Big|\prod_{n<m}P_n\Big|\left(1-\frac{1}{k_m}\Big|\sum_{s=1}^{k_m}e^{2\pi i s\theta}\Big|\right)^2\right\}^2.$$

Since

$$\int\Big|\prod_{n<m}P_n\Big|\left(\frac{1}{k_m}\Big|\sum_{s=1}^{k_m}e^{2\pi i s\theta}\Big|\right) < k_m^{-\frac12}\Big(\int\Big|\prod P_n\Big|^2\Big)^{\frac12} \le k_m^{-\frac12} \tag{2.31}$$

also

$$\int\Big|\prod_{n<m}P_n\cdot P_m\Big| \le \int\Big|\prod_{n<m}P_n\Big| - c\Big(\int\Big|\prod_{n<m}P_n\Big|\Big)^2 + 2k_m^{-\frac12}, \tag{2.32}$$

which immediately implies (2.4) since $\Sigma k_m^{-\frac12} < \infty$.

This concludes the proof of Proposition 1.

B. Weiss pointed out the following corollary to the author.

PROPOSITION 2: *The Ornstein rank 1 transformations are mixing of any order.*

Proof: According to a recent result of B. Host [2], any mixing transformation with singular spectral type has this property.

Remark: It is a well-known open problem whether mixing transformations are mixing of all order. In the case of rank 1 transformations (in general) S. Kalikow [3] proved that 2-mixing implies 3-mixing. His proof is based on different methods.

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* See [1], for instance.

3. Appendix

A natural problem suggested by the preceding (if one tries to prove that the maximal spectral type of any rank 1 transformation is singular) is the following question about trigonometric polynomials.

Define for $n > 1$

$$\beta_n = \sup_{k_1<k_2<\cdots<k_n} \frac{1}{\sqrt{n}} \left\| \sum_{j=1}^{n} e^{2\pi i k_j \theta} \right\|_{L^1(\Pi)} \tag{3.1}$$

Is $\sup_{n>1} \beta_n < 1$?

At the time of this writing, the best I know to do on this question is contained in following.

PROPOSITION 2:

$$\beta_n < 1 - c\frac{\log n}{n}. \tag{3.2}$$

Proof: Define $f = (1/\sqrt{n}) \sum_{j=1}^{n} e^{2\pi i k_j \theta}$. One has

$$\frac{1}{2} < \int \left|1 - |f|^2\right|^2 = \int \left|1 - |f|\right|^2 \left|1 + |f|\right|^2 \leq 2(1+\sqrt{n})^2 \left(1 - \int |f|\right) \tag{3.3}$$

so that at least

$$1 - \int |f| > \frac{c}{n}.$$

Also

$$1 - \int |f| = \frac{1}{2} \int (1 - |f|)^2 \geq \frac{1}{2} \int_{|f|>10} (|f| - 1)^2 > \frac{1}{4} \int_{|f|>10} \left| \, |f|^2 - 1 \right|. \tag{3.4}$$

Write

$$|f|^2 - 1 = \frac{1}{n} \sum_{\ell \neq 0} d(\ell) e^{2\pi i \ell \theta} \quad \text{where } d(\ell) = \#\{(j,j') \mid k_j - k_{j'} = \ell\}. \tag{3.5}$$

One has clearly

$$n^2 - n = \sum_{\ell \neq 0} d(\ell) \leq |S|^{\frac{1}{2}} \left(\sum \left| d(\ell) \right|^2 \right)^{\frac{1}{2}} < n|S|^{\frac{1}{2}} \| 1 - |f|^2 \|_2 \tag{3.6}$$

denoting

$$S = \{\ell \in \mathbb{Z} | d(\ell) \neq 0\}.$$

At this point, we claim for a fixed integer $r < c \log |S|$ the existence of a trigonometric polynomial φ with following properties:

$$\|\varphi\|_\infty \leq 1, \tag{3.7}$$

$$\hat{\varphi} > 0 \quad \text{on } S, \tag{3.8}$$

$$\sum_{\ell \in S} \hat{\varphi}(\ell) > r, \tag{3.9}$$

$$\|\varphi\|_2 < \frac{c^r}{|S|^{\frac{1}{2}}}. \tag{3.10}$$

The construction of φ is a variant of the McGehee–Pigno–Smith construction in their proof of the Littlewood conjecture (see [4]). The only difference is that their number of "steps" is $\sim \log |S|$, while here we limit ourselves to r steps. Property (3.10) is immediate from the way φ is built in [4].

From (3.8), (3.9)

$$\langle |f|^2 - 1, \varphi \rangle > \frac{1}{n} r. \tag{3.11}$$

Also from (3.10)

$$\int_{|f|<10} \big| |f|^2 - 1 \big| \, |\varphi| \leq 11 \|1 - |f|\|_2 \, \|\varphi\|_2 < \frac{c^r}{|S|^{\frac{1}{2}}} \|1 - |f| \,\|_2. \tag{3.12}$$

Assuring $1 - \int |f| < (\log n)/n$, one finds

$$\|1 - |f|\|_2 < \left(\frac{\log n}{n}\right)^{\frac{1}{2}}.$$

Also, by (3.3)

$$\int \left|1 - |f|^2\right|^2 < c \log n \tag{3.13}$$

which implies, by (3.6),

$$|S| > c \frac{n^2}{\log n}. \tag{3.14}$$

Substitution of (3.13), (3.14) in (3.12) gives the bound $(c^r / n\sqrt{n}) \log n$. This quantity will be small with respect to (3.11) provided $c^r << \sqrt{n} / \log n$, permitting to let $r \sim \log n$. Consequently

$$\int_{|f|>10} \big| |f|^2 - 1 \big| \, |\varphi| > (1 - o(1)) \frac{r}{n} > c \frac{\log n}{n}$$

which completes the proof, because of (3.4), (3.7).

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